Simple Type Theory as Framework for Combining Logics

Christoph Benzmüller*

Articulate Software, Angwin, CA, U.S.

Abstract. Simple type theory is suited as framework for combining classical and non-classical logics. This claim is based on the observation that various prominent logics, including (quantified) multimodal logics and intuitionistic logics, can be elegantly embedded in simple type theory. Furthermore, simple type theory is sufficiently expressive to model combinations of embedded logics and it has a well understood semantics. Off-the-shelf reasoning systems for simple type theory exist that can be uniformly employed for reasoning *within* and *about* combinations of logics.

1 Introduction

Church's simple type theory $\mathcal{STT}$ [14], also known as classical higher-order logic, is suited as a framework for combining classical and non-classical logics. This is what this paper illustrates.

Evidently, $\mathcal{STT}$ has many prominent classical logic fragments, including propositional and first-order logic, the guarded fragment, second-order logic, monadic second-order logic, the basic fragment of $\mathcal{STT}$, etc. Interestingly, also prominent non-classical logics – including quantified multi-modal logics and intuitionistic logic – can be elegantly embedded in $\mathcal{STT}$. It is thus not surprising that also combinations of such logics can be flexibly modeled within $\mathcal{STT}$. Our claim is furthermore supported by the fact that the semantics of $\mathcal{STT}$ is well understood [1,2,7,22] and that powerful proof assistants and automated theorem provers for $\mathcal{STT}$ already exist. The automation of $\mathcal{STT}$ currently experiences a renaissance that has been fostered by the recent extension of the successful TPTP infrastructure for first-order logic [29] to higher-order logic, called TPTP THF [11,30]. Exploiting this new infrastructure we will demonstrate how higher-order automated theorem provers and model generators can be employed for reasoning *within* and *about* combinations of logics.

In Sect. 2 we outline our embedding of quantified multimodal logics in $\mathcal{STT}$. Further logic embeddings in $\mathcal{STT}$ are discussed in Sect. 3; our examples comprise intuitionistic logic, access control logics and the region connection calculus. In Sect. 4 we illustrate how the reasoning *about* logics and their combinations is

* This work has been funded by the German Research Foundation (DFG) under grant BE 2501/6-1.

facilitated in our approach, and in Sect. 5 we employ simple examples to demonstrate the application of our approach for reasoning *within* combined logics. The performance results of our experiments with off-the-shelf, TPTP THF compliant higher-order automated reasoning systems are presented in Sect. 6.

2 (Normal) Quantified Multimodal Logics in $\mathcal{STT}$

$\mathcal{STT}$ [14] is based on the simply typed λ-calculus. The set $\mathcal{T}$ of simple types is usually freely generated from a set of basic types $\{o, \iota\}$ (where o is the type of Booleans and ι is the type of individuals) using the right-associative function type constructor $\rightarrow$. Instead of $\{o, \iota\}$ we here consider a set of base types $\{o, \iota, \mu\}$, providing an additional base type μ (the type of possible worlds).

The simple type theory language $\mathcal{STT}$ is defined by (where α, β, $o \in \mathcal{T}$):

$$\begin{aligned} s, t ::= {} & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha . s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta}\, t_\alpha)_\beta \mid (\neg_{o \rightarrow o}\, s_o)_o \mid \\ & (s_o \vee_{o \rightarrow o \rightarrow o} t_o)_o \mid (s_\alpha =_{\alpha \rightarrow \alpha \rightarrow o} t_\alpha)_o \mid (\Pi_{(\alpha \rightarrow o) \rightarrow o}\, s_{\alpha \rightarrow o})_o \end{aligned}$$

p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. Our logical connectives of choice are $\neg_{o \rightarrow o}$, $\vee_{o \rightarrow o \rightarrow o}$, $=_{\alpha \rightarrow \alpha \rightarrow o}$ and $\Pi_{(\alpha \rightarrow o) \rightarrow o}$ (for each type α).[1] From these connectives, other logical connectives can be defined in the usual way (e.g., $\wedge$ and $\Rightarrow$). We often use binder notation $\forall X_\alpha . s$ for $\Pi_{(\alpha \rightarrow o) \rightarrow o}(\lambda X_\alpha . s_o)$. We assume familiarity with α-conversion, β- and η-reduction, and the existence of β- and $\beta\eta$-normal forms. Moreover, we obey the usual definitions of free variable occurrences and substitutions.

The semantics of $\mathcal{STT}$ is well understood and thoroughly documented in the literature [1,2,7,22]. The semantics of choice for our work is Henkin semantics.

Quantified modal logics have been studied by Fitting [15] (further related work is available by Blackburn and Marx [12] and Braüner [13]). In contrast to Fitting we are here not interested only in **S5** structures but in the more general case of **K** from which more constrained structures (such as **S5**) can be easily obtained. First-order quantification can be constant domain or varying domain. Below we only consider the constant domain case: every possible world has the same domain. Like Fitting, we keep our definitions simple by not having function or constant symbols. While Fitting [15] studies quantified monomodal logic, we are interested in quantified multimodal logic. Hence, we introduce multiple $\Box_r$ operators for symbols r from an index set S. The grammar for our quantified multimodal logic $\mathcal{QML}$ hence is

$$s, t ::= P \mid k(X^1, \ldots, X^n) \mid \neg\, s \mid s \vee t \mid \forall X . s \mid \forall P . s \mid \Box_r\, s$$

where P denotes propositional variables, X, X^i denote first-order (individual) variables, and k denotes predicate symbols of any arity. Further connectives,

[1] This choice is not minimal (from $=_{\alpha \rightarrow \alpha \rightarrow o}$ all other logical constants can already be defined [3]). It useful though in the context of resolution based theorem proving.

quantifiers, and modal operators can be defined as usual. We also obey the usual definitions of free variable occurrences and substitutions.

Fitting introduces three different notions of Kripke semantics for $\mathcal{QML}$: $\mathbf{QS5}\pi^-$, $\mathbf{QS5}\pi$, and $\mathbf{QS5}\pi^+$. In our work [9] we study related notions $\mathbf{QK}\pi^-$, $\mathbf{QK}\pi$, and $\mathbf{QK}\pi^+$ for a modal context $\mathbf{K}$, and we support multiple modalities.

$\mathcal{STT}$ is an expressive logic and it is thus not surprising that $\mathcal{QML}$ can be elegantly modeled and even automated as a fragment of $\mathcal{STT}$. The idea of the encoding, called $\mathcal{QML}^{STT}$, is simple. Choose type ι to denote the (non-empty) set of individuals and we reserve a second base type μ to denote the (non-empty) set of possible worlds. The type o denotes the set of truth values. Certain formulas of type $\mu \to o$ then correspond to multimodal logic expressions. The multimodal connectives $\neg$, $\vee$, and $\Box$, become λ-terms of types $(\mu \to o) \to (\mu \to o)$, $(\mu \to o) \to (\mu \to o) \to (\mu \to o)$, and $(\mu \to \mu \to o) \to (\mu \to o) \to (\mu \to o)$ respectively.

Quantification is handled as in $\mathcal{STT}$ by modeling $\forall X_\blacksquare p$ as $\Pi(\lambda X_\blacksquare .p)$ for a suitably chosen connective Π. Here we are interested in defining two particular modal $\boldsymbol{\Pi}$-connectives: $\boldsymbol{\Pi}^\iota$, for quantification over individual variables, and $\boldsymbol{\Pi}^{\mu\to o}$, for quantification over modal propositional variables that depend on worlds. They become terms of type $(\iota \to (\mu \to o)) \to (\mu \to o)$ and $((\mu \to o) \to (\mu \to o)) \to (\mu \to o)$ respectively.

The $\mathcal{QML}^{STT}$ modal operators $\neg, \vee, \Box, \boldsymbol{\Pi}^\iota$, and $\boldsymbol{\Pi}^{\mu\to o}$ are now simply defined as follows:

$$
\begin{aligned}
\neg_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}{}_\blacksquare \lambda W_{\mu}{}_\blacksquare \neg\phi\, W \\
\vee_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}{}_\blacksquare \lambda\psi_{\mu\to o}{}_\blacksquare \lambda W_{\mu}{}_\blacksquare \phi\, W \vee \psi\, W \\
\Box_{(\mu\to\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda R_{\mu\to\mu\to o}{}_\blacksquare \lambda\phi_{\mu\to o}{}_\blacksquare \lambda W_{\mu}{}_\blacksquare \forall V_{\mu}{}_\blacksquare \neg R\, W\, V \vee \phi\, V \\
\boldsymbol{\Pi}^\iota_{(\iota\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{\iota\to(\mu\to o)}{}_\blacksquare \lambda W_{\mu}{}_\blacksquare \forall X_{\iota}{}_\blacksquare \phi\, X\, W \\
\boldsymbol{\Pi}^{\mu\to o}_{((\mu\to o)\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{(\mu\to o)\to(\mu\to o)}{}_\blacksquare \lambda W_{\mu}{}_\blacksquare \forall P_{\mu\to o}{}_\blacksquare \phi\, P\, W
\end{aligned}
$$

Note that our encoding actually only employs the second-order fragment of $\mathcal{STT}$ enhanced with lambda-abstraction.

Further operators can be introduced as usual, for example, $\top = \lambda W_{\mu}{}_\blacksquare \top$, $\bot = \neg\,\top$, $\wedge = \lambda\phi,\psi_\blacksquare \neg\,(\neg\,\phi \vee \neg\,\psi)$, $\supset = \lambda\phi,\psi_\blacksquare \neg\,\phi \vee \psi$, $\Leftrightarrow = \lambda\phi,\psi_\blacksquare (\phi \supset \psi) \wedge (\psi \supset \phi)$, $\Diamond = \lambda R,\phi_\blacksquare \neg\,(\Box\, R\,(\neg\,\phi))$, $\boldsymbol{\Sigma}^\iota = \lambda\phi_\blacksquare \neg\,\boldsymbol{\Pi}^\iota(\lambda X_\blacksquare \neg\,\phi\, X)$, $\boldsymbol{\Sigma}^{\mu\to o} = \lambda\phi_\blacksquare \neg\,\boldsymbol{\Pi}^{\mu\to o}(\lambda P_\blacksquare \neg\,\phi\, P)$.

For defining $\mathcal{QML}^{STT}$-propositions we fix a set $\mathcal{IV}^{STT}$ of individual variables of type ι, a set $\mathcal{PV}^{STT}$ of propositional variables[2] of type $\mu \to o$, and a set $\mathcal{SYM}^{STT}$ of n-ary (curried) predicate constants of types $\underbrace{\iota \to \ldots \to \iota}_{n} \to (\mu \to o)$. Moreover, we fix a set $\mathcal{S}^{STT}$ of accessibility relation constants of type $\mu \to \mu \to o$. $\mathcal{QML}^{STT}$-propositions are now defined as the smallest set of $\mathcal{STT}$-terms for which the following hold:

– if $P \in \mathcal{PV}^{STT}$, then $P \in \mathcal{QML}^{STT}$

[2] Note that the denotation of propositional variables depends on worlds.

– if $X^j \in \mathcal{IV}^{STT}$ $(j = 1, \ldots, n)$ and $k \in \mathcal{SYM}^{STT}$, then $(k\, X^1 \ldots X^n) \in \mathcal{QML}^{STT}$
– if $\phi, \psi \in \mathcal{QML}^{STT}$, then $\neg\, \phi \in \mathcal{QML}^{STT}$ and $\phi \vee \psi \in \mathcal{QML}^{STT}$
– if $r \in \mathcal{S}^{STT}$ and $\phi \in \mathcal{QML}^{STT}$, then $\Box\, r\, \phi \in \mathcal{QML}^{STT}$.
– if $X \in \mathcal{IV}^{STT}$ and $\phi \in \mathcal{QML}^{STT}$, then $\boldsymbol{\Pi}^{\iota}(\lambda X_\centerdot \phi) \in \mathcal{QML}^{STT}$
– if $P \in \mathcal{PV}^{STT}$ and $\phi \in \mathcal{QML}^{STT}$, then $\boldsymbol{\Pi}^{\mu \to o}(\lambda P_\centerdot \phi) \in \mathcal{QML}^{STT}$

We write $\Box_r\, \phi$ for $\Box\, r\, \phi$, $\forall X_{\iota}{}_\centerdot \phi$ for $\boldsymbol{\Pi}^{\iota}(\lambda X_{\iota}{}_\centerdot \phi)$, and $\forall P_{\mu \to o}{}_\centerdot \phi$ for $\boldsymbol{\Pi}^{\mu \to o}(\lambda P_{\mu \to o}{}_\centerdot \phi)$.

Note that the defining equations for our $\mathcal{QML}$ modal operators are themselves formulas in $\mathcal{STT}$. Hence, we can express $\mathcal{QML}$ formulas in a higher-order reasoner elegantly in the usual syntax. For example, $\Box_r\, \exists P_{\mu \to o}{}_\centerdot\, P$ is a $\mathcal{QML}^{STT}$ proposition; it has type $\mu \to o$.

Validity of $\mathcal{QML}^{STT}$ propositions is defined in the obvious way: a $\mathcal{QML}$-proposition $\phi_{\mu \to o}$ is valid if and only if for all possible worlds w_μ we have $w \in \phi_{\mu \to o}$, that is, if and only if $\phi_{\mu \to o}\, w_\mu$ holds. Hence, the notion of validity is modeled via the following equation (alternatively we could define valid simply as $\Pi_{(\mu \to o) \to o}$):

$$\text{valid} = \lambda \phi_{\mu \to o}{}_\centerdot \forall W_{\mu}{}_\centerdot\, \phi\, W$$

Now we can formulate proof problems in $\mathcal{QML}^{STT}$, e.g., valid $\Box_r\, \exists P_{\mu \to o}{}_\centerdot\, P$. Using rewriting or definition expanding, we can reduce such proof problems to corresponding statements containing only the basic connectives $\neg$, $\vee$, $=$, Π^{ι}, and $\Pi^{\mu \to o}$ of $\mathcal{STT}$. In contrast to the many other approaches no external transformation mechanism is required. For our example formula valid $\Box_r\, \exists P_{\mu \to o}{}_\centerdot\, P$ unfolding and $\beta\eta$-reduction leads to $\forall W_{\mu}{}_\centerdot \forall Y_{\mu}{}_\centerdot \neg r\, W\, Y \vee (\neg \forall X_{\mu \to o}{}_\centerdot \neg (X\, Y))$. It is easy to check that this formula is valid in Henkin semantics: put $X = \lambda Y_{\mu}{}_\centerdot \top$.

We have proved soundness and completeness for this embedding [9], that is, for $s \in \mathcal{QML}$ and the corresponding $s_{\mu \to o} \in \mathcal{QML}^{STT} \subset \mathcal{STT}$ we have:

Theorem 1. $\models^{\mathcal{STT}}$ *(valid* $s_{\mu \to o}$*) if and only if* $\models^{\mathbf{QK}\pi}$ s.

This result also illustrates the correspondence between $\mathbf{QK}\pi$ models and Henkin models; for more details see [9].

Obviously, the reduction of our embedding to first-order multimodal logics (which only allow quantification over individual variables), to propositional quantified multimodal logics (which only allow quantification over propositional variables) and to propositional multimodal logics (no quantifiers) is sound and complete. Extending our embedding for hybrid logics is straightforward [23]; note in particular that denomination of individual worlds using constant symbols of type μ is easily possible.

In the remainder we will often omit type information. It is sufficient to remember that worlds are of type μ, multimodal propositions of type $\mu \to o$, and accessibility relations of type $\mu \to \mu \to o$. Individuals are of type ι.

3 Embeddings of Other Logics in $\mathcal{STT}$

We have studied several other logic embeddings in $\mathcal{STT}$, some of which will be mentioned in this section.

Intuitionistic Logics Gödels interpretation of propositional intuitionistic logic in propositional modal logic $S4$ [19] can be combined with our results from the previous section in order to provide a sound and complete embedding of propositional intuitionistic logic into $\mathcal{STT}$ [9].

Gödel studies the propositional intuitionistic logic $\mathcal{IPL}$ defined by

$$s, t ::= p \mid \dot{\neg}\, s \mid s \,\dot{\supset}\, t \mid s \,\dot{\vee}\, t \mid p \,\dot{\wedge}\, t$$

He introduces the a mapping from $\mathcal{IPL}$ into propositional modal logic $S4$ which maps $\dot{\neg}\, s$ to $\neg\, \Box_r\, s$, $s \,\dot{\supset}\, t$ to $\Box_r\, s \supset \Box_r\, t$, $s \,\dot{\vee}\, t$ to $\Box_r\, s \vee \Box_r\, t$, and $s \,\dot{\wedge}\, t$ to $s \wedge t$.[3] By simply combining Gödel's mapping with our mapping from before we obtain the following embedding of $\mathcal{IPL}$ in $\mathcal{STT}$.

Let $\mathcal{IPL}$ be a propositional intuitionistic logic with atomic primitives p^1, $\ldots$, p^m $(m \geq 1)$. We define the set $\mathcal{IPL}^{\mathcal{STT}}$ of corresponding propositional intuitionistic logic propositions in $\mathcal{STT}$ as follows.

1. For the atomic $\mathcal{IPL}$ primitives p^1, $\ldots$, p^m we introduce corresponding $\mathcal{IPL}^{\mathcal{STT}}$ predicate constants $p^1_{\mu\to o}, \ldots, p^m_{\mu\to o}$. Moreover, we provide the single accessibility relation constant $r_{\mu\to\mu\to o}$.
2. Corresponding to Gödel's mapping we introduce the logical connectives of $\mathcal{IPL}^{\mathcal{STT}}$ as abbreviations for the following λ-terms (we omit the types here):

$$\begin{aligned}
\dot{\neg} &= \lambda\phi_{\blacksquare}\lambda W_{\blacksquare}\neg\forall V_{\blacksquare}\neg r\, W\, V \vee \phi\, V\\
\dot{\supset} &= \lambda\phi_{\blacksquare}\lambda\psi_{\blacksquare}\lambda W_{\blacksquare}\neg(\forall V_{\blacksquare}\neg r\, W\, V \vee \phi\, V) \vee (\forall V_{\blacksquare}\neg r\, W\, V \vee \psi\, V)\\
\dot{\vee} &= \lambda\phi_{\blacksquare}\lambda\psi_{\blacksquare}\lambda W_{\blacksquare}(\forall V_{\blacksquare}\neg r\, W\, V \vee \phi\, V) \vee (\forall V_{\blacksquare}\neg r\, W\, V \vee \psi\, V)\\
\dot{\wedge} &= \lambda\phi_{\blacksquare}\lambda\psi_{\blacksquare}\lambda W_{\blacksquare}\neg(\neg\phi\, W \vee \neg\psi\, W)
\end{aligned}$$

3. We define the set of $\mathcal{IPL}^{\mathcal{STT}}$-propositions as the smallest set of simply typed λ-terms for which the following hold:
   - $p^1_{\mu\to o}, \ldots, p^m_{\mu\to o}$ define the atomic $\mathcal{IPL}^{\mathcal{STT}}$-propositions.
   - If ϕ and ψ are $\mathcal{IPL}^{\mathcal{STT}}$-propositions, then so are $\dot{\neg}\, \phi$, $\phi \,\dot{\supset}\, \psi$, $\phi \,\dot{\vee}\, \psi$, and $\phi \,\dot{\wedge}\, \psi$.

The notion of validity we adopt is the same as for $\mathcal{QML}^{STT}$. However, since Gödel connects $\mathcal{IPL}$ with modal logic $S4$, we transform each proof problem $t \in \mathcal{IPL}$ into a corresponding proof problem t' in $\mathcal{STT}$ of the following form

$$t' := ((\text{valid}\, \forall\phi_{\mu\to o\blacksquare}\, \Box_r\, \phi \supset \phi) \wedge (\text{valid}\, \forall\phi_{\mu\to o\blacksquare}\, \Box_r\, \phi \supset \Box_r\, \Box_r\, \phi)) \Rightarrow (\text{valid}\, t_{\mu\to o})$$

where $t_{\mu\to o}$ is the $\mathcal{IPL}^{\mathcal{STT}}$ term for t according to our definition above. Alternatively we may translate t into $t'' := ((\text{reflexive}\, r) \wedge (\text{transitive}\, r)) \Rightarrow (\text{valid}\, t_{\mu\to o})$.

Combining soundness [19] and completeness [24] of Gödel's embedding with Theorem 1 we obtain the following soundness and completeness result: Let $t \in \mathcal{IPL}$ and let $t' \in \mathcal{STT}$ as constructed above. t is valid in propositional intuitionistic logic if and only if t' is valid in $\mathcal{STT}$.

[3] Alternative mappings have been proposed and studied in the literature which we could employ here equally as well.

Example problems in intuitionistic logic have been encoded in THF syntax [11] and added to the TPTP THF library[4] and are accessible under identifiers SYO058ˆ4 – SYO074ˆ4.

Access Control Logics Garg and Abadi recently translated several prominent access control logics into modal logic S4 and proved these translations sound and complete [17]. We have combined this work with our above results in order to obtain a sound and complete embedding of these access control logics in $\mathcal{STT}$ and we have carried out experiments with the prover LEO-II [6]. Example problems have been added to the TPTP THF library and are accessible under identifiers SWV425ˆx – SWV436ˆx (for $x \in \{1, \ldots, 4\}$).

Logics for Spatial Reasoning Evidently, the region connection calculus [26] is a fragment of $\mathcal{STT}$: choose a base type r ('region') and a reflexive and symmetric relation c ('connected') of type $r \to r \to o$ and define (where X, Y, and Z are variables of type r):

$$
\begin{aligned}
\text{disconnected}: \quad dc &= \lambda X, Y_\blacksquare \neg(c\ X\ Y) \\
\text{part of}: \quad p &= \lambda X, Y_\blacksquare \forall Z_\blacksquare ((c\ Z\ X) \Rightarrow (c\ Z\ Y)) \\
\text{identical with}: \quad eq &= \lambda X, Y_\blacksquare ((p\ X\ Y) \wedge (p\ Y\ X)) \\
\text{overlaps}: \quad o &= \lambda X, Y_\blacksquare \exists Z_\blacksquare ((p\ Z\ X) \wedge (p\ Z\ Y)) \\
\text{partially overlaps}: \quad po &= \lambda X, Y_\blacksquare ((o\ X\ Y) \wedge \neg(p\ X\ Y) \wedge \neg(p\ Y\ X)) \\
\text{externally connected}: \quad ec &= \lambda X, Y_\blacksquare ((c\ X\ Y) \wedge \neg(o\ X\ Y)) \\
\text{proper part}: \quad pp &= \lambda X, Y_\blacksquare ((p\ X\ Y) \wedge \neg(p\ Y\ X)) \\
\text{tangential proper part}: \quad tpp &= \lambda X, Y_\blacksquare ((pp\ X\ Y) \wedge \exists Z_\blacksquare ((ec\ Z\ X) \wedge (ec\ Z\ Y))) \\
\text{nontang. proper part}: \quad ntpp &= \lambda X, Y_\blacksquare ((pp\ X\ Y) \wedge \neg\exists Z_\blacksquare ((ec\ Z\ X) \wedge (ec\ Z\ Y)))
\end{aligned}
$$

An example problem for the region connection calculus will be discussed below.

4 Reasoning about Logics and Combinations of Logics

We illustrate how our approach supports reasoning about logics and their combinations. First, we focus on modal logics and their well known relationships between properties of accessibility relations and corresponding modal axioms (respectively axiom schemata) [21]. Such meta-theoretic insights can be elegantly encoded (and, as we will later see, automatically proved) in our approach. First

[4] TPTP THF problems for various problem categories are available at `http://www.cs.miami.edu/~tptp/cgi-bin/SeeTPTP?Category=Problems`; all problem identifiers with an 'ˆ' in their name refer to higher-order THF problems. The TPTP library meanwhile contains more than 2700 example problems in THF syntax.

we encode various accessibility relation properties in $\mathcal{STT}$:

$$\begin{aligned}
\text{reflexive} &= \lambda R_\blacksquare \forall S_\blacksquare R\, S\, S && (1)\\
\text{symmetric} &= \lambda R_\blacksquare \forall S, T_\blacksquare ((R\, S\, T) \Rightarrow (R\, T\, S)) && (2)\\
\text{serial} &= \lambda R_\blacksquare \forall S_\blacksquare \exists T_\blacksquare (R\, S\, T) && (3)\\
\text{transitive} &= \lambda R_\blacksquare \forall S, T, U_\blacksquare ((R\, S\, T) \wedge (R\, T\, U) \Rightarrow (R\, S\, U)) && (4)\\
\text{euclidean} &= \lambda R_\blacksquare \forall S, T, U_\blacksquare ((R\, S\, T) \wedge (R\, S\, U) \Rightarrow (R\, T\, U)) && (5)\\
\text{partially_functional} &= \lambda R_\blacksquare \forall S, T, U_\blacksquare ((R\, S\, T) \wedge (R\, S\, U) \Rightarrow T = U) && (6)\\
\text{functional} &= \lambda R_\blacksquare \forall S_\blacksquare \exists T_\blacksquare ((R\, S\, T) \wedge \forall U_\blacksquare ((R\, S\, U) \Rightarrow T = U)) && (7)\\
\text{weakly_dense} &= \lambda R_\blacksquare \forall S, T_\blacksquare ((R\, S\, T) \Rightarrow \exists U_\blacksquare ((R\, S\, U) \wedge (R\, U\, T))) && (8)\\
\text{weakly_connected} &= \lambda R_\blacksquare \forall S, T, U_\blacksquare (((R\, S\, T) \wedge (R\, S\, U)) \Rightarrow \\
&\qquad ((R\, T\, U) \vee T = U \vee (R\, U\, T))) && (9)\\
\text{weakly_directed} &= \lambda R_\blacksquare \forall S, T, U_\blacksquare (((R\, S\, T) \wedge (R\, S\, U)) \Rightarrow \\
&\qquad \exists V_\blacksquare ((R\, T\, V) \wedge (R\, U\, V))) && (10)
\end{aligned}$$

Remember, that R is of type $\mu \to \mu \to o$ and S, T, U are of type μ. The corresponding axioms are given next.

$$\begin{aligned}
M &: \forall \phi_\blacksquare \Box_r\, \phi \supset \phi && (11)\\
B &: \forall \phi_\blacksquare \phi \supset \Box_r \Diamond_r\, \phi && (12)\\
D &: \forall \phi_\blacksquare \Box_r\, \phi \supset \Diamond_r\, \phi && (13)\\
4 &: \forall \phi_\blacksquare \Box_r\, \phi \supset \Box_r \Box_r\, \phi && (14)\\
5 &: \forall \phi_\blacksquare \Diamond_r\, \phi \supset \Box_r \Diamond_r\, \phi && (15)
\end{aligned}$$

$$\begin{aligned}
&\forall \phi_\blacksquare \Diamond_r\, \phi \supset \Box_r\, \phi && (16)\\
&\forall \phi_\blacksquare \Diamond_r\, \phi \Leftrightarrow \Box_r\, \phi && (17)\\
&\forall \phi_\blacksquare \Box_r \Box_r\, \phi \supset \Box_r\, \phi && (18)\\
&\forall \phi, \psi_\blacksquare \Box_r\, ((\phi \wedge \Box_r\, \phi) \supset \psi) \vee \\
&\qquad \Box_r\, ((\psi \wedge \Box_r\, \psi) \supset \phi) && (19)\\
&\forall \phi_\blacksquare \Diamond_r \Box_r\, \phi \supset \Box_r \Diamond_r\, \phi && (20)
\end{aligned}$$

Example 1. For k ($k = (1), \ldots, (10)$) we can now easily formulate the well known correspondence theorems $(k) \Rightarrow (k + 10)$ and $(k) \Leftarrow (k + 10)$. For example,

$$(1) \Rightarrow (11) : \quad \forall R_\blacksquare (\text{reflexive } R) \Rightarrow (\text{valid } \forall \phi_\blacksquare \Box_R\, \phi \supset \phi)$$

Example 2. There are well known relationships between different modal logics and there exist alternatives for their axiomatization (cf. the relationship map in [18]). For example, for modal logic S5 we may choose axioms M and 5 as standard axioms. Respectively for logic KB5 we may choose B and 5. We may then want to investigate the following conjectures (the only one that does not hold is (31)):

$$
\begin{aligned}
S5 = M5 &\Leftrightarrow MB5 && (21)\\
&\Leftrightarrow M4B5 && (22)\\
&\Leftrightarrow M45 && (23)\\
&\Leftrightarrow M4B && (24)\\
&\Leftrightarrow D4B && (25)\\
&\Leftrightarrow D4B5 && (26)\\
&\Leftrightarrow DB5 && (27)
\end{aligned}
$$

$$
\begin{aligned}
KB5 &\Leftrightarrow K4B5 && (28)\\
&\Leftrightarrow K4B && (29)\\
\\
M5 &\Rightarrow D45 && (30)\\
D45 &\Rightarrow M5 && (31)
\end{aligned}
$$

Exploiting the correlations $(k) \Leftrightarrow (k + 10)$ from before these problems can be formulated as follows; we give the case for M5 $\Leftrightarrow$ D4B:

$$\forall R\text{.}(((\text{reflexive } R) \wedge (\text{euclidean } R)) \Leftrightarrow ((\text{serial } R) \wedge (\text{transitive } R) \wedge (\text{symmetric } R)))$$

Example 3. We can also encode the Barcan formula and its converse. (They are theorems in our approach, which confirms that we are 'constant domain'.)

$$
\begin{aligned}
BF: &\quad \text{valid } \forall X_\iota\text{.}\, \Box_r \, (p_{\iota\to(\mu\to o)} \, X) \supset \Box_r \, \forall X_\iota\text{.}\, (p_{\iota\to(\mu\to o)} \, X) && (32)\\
BF^{-1}: &\quad \text{valid } \Box_r \, \forall X_\iota\text{.}\, (p_{\iota\to(\mu\to o)} \, X) \supset \forall X_\iota\text{.}\, \Box_r \, (p_{\iota\to(\mu\to o)} \, X) && (33)
\end{aligned}
$$

Example 4. An interesting meta property for combined logics with modalities $\Diamond_i, \Box_j, \Box_k$, and $\Diamond_l$ is the correspondence between the following axiom and the (i, j, k, l)-confluence property

$$
\begin{aligned}
&(\text{valid } \forall \phi\text{.}\, (\Diamond_i \, \Box_j \, \phi) \supset \Box_k \, \Diamond_l \, \phi)\\
\Leftrightarrow\ &(\forall A\text{.}\forall B\text{.}\forall C\text{.}(((i\, A\, B) \wedge (k\, A\, C)) \Rightarrow \exists D\text{.}((j\, B\, D) \wedge (l\, C\, D)))) \qquad (34)
\end{aligned}
$$

Example 5. Segerberg [27] discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context. This statement can be encoded in our framework as follows:

$$
\begin{aligned}
&(\text{reflexive } a), (\text{transitive } a), (\text{euclid. } a), (\text{reflexive } b), (\text{transitive } b), (\text{euclid. } b),\\
&(\mathit{valid}\, \forall \phi\text{.}\, \Box_a \, \Box_b \, \phi \Leftrightarrow \Box_b \, \Box_a \, \phi)\\
&\models^{\mathcal{STT}} \ (\mathit{valid}\, \forall \phi, \psi\text{.}\, \Box_a \, (\Box_a \, \phi \vee \Box_b \, \psi) \supset (\Box_a \, \phi \vee \Box_a \, \psi)) \wedge\\
&\qquad\quad (\mathit{valid}\, \forall \phi, \psi\text{.}\, \Box_b \, (\Box_a \, \phi \vee \Box_b \, \psi) \supset (\Box_b \, \phi \vee \Box_b \, \psi)) \qquad (35)
\end{aligned}
$$

Example 6. Suppose we want to work with a 2-dimensional logic combining a modality $\Box_k$ of knowledge with a modality $\Box_b$ of belief. Moreover, suppose we model $\Box_k$ as an S5 modality and $\Box_b$ as an D45 modality and let us furthermore add two axioms characterizing their relationship. We may then want to check whether or not $\Box_k$ and $\Box_b$ coincide, i.e., whether $\Box_k$ includes $\Box_b$:

$$
\begin{aligned}
&(\text{reflexive } k), (\text{transitive } k), (\text{euclid. } k), (\text{serial } b), (\text{transitive } b), (\text{euclid. } b),\\
&(\mathit{valid}\, \forall \phi\text{.}\, \Box_k \, \phi \supset \Box_b \, \phi), (\mathit{valid}\, \forall \phi\text{.}\, \Box_b \, \phi \supset \Box_b \, \Box_k \, \phi)\\
&\models^{\mathcal{STT}} \ (\mathit{valid}\, \forall \phi\text{.}\, \Box_b \, \phi \supset \Box_k \, \phi) \qquad (36)
\end{aligned}
$$

5 Reasoning within Combined Logics

We illustrate how our approach supports reasoning within combined logics. First we present two examples in epistemic reasoning. Our formulation in both cases adapts Baldoni's modeling [5].

Example 7 (Epistemic reasoning: The friends puzzle). (i) Peter is a friend of John, so if Peter knows that John knows something then John knows that Peter knows the same thing. (ii) Peter is married, so if Peter's wife knows something, then Peter knows the same thing. John and Peter have an appointment, let us consider the following situation: (a) Peter knows the time of their appointment. (b) Peter also knows that John knows the place of their appointment. Moreover, (c) Peter's wife knows that if Peter knows the time of their appointment, then John knows that too (since John and Peter are friends). Finally, (d) Peter knows that if John knows the place and the time of their appointment, then John knows that he has an appointment. From this situation we want to prove (e) that each of the two friends knows that the other one knows that he has an appointment.

For modeling the knowledge of Peter, Peter's wife, and John we consider a 3-dimensional logic combining the modalities $\Box_{\mathrm{p}}$, $\Box_{(\mathrm{w\,p})}$, and $\Box_{\mathrm{j}}$. Actually modeling them as S4 modalities turns out to be sufficient for this example. Hence, we introduce three corresponding accessibility relations j, p, and (w p). The S4 axioms for $x \in \{\mathrm{j}, \mathrm{p}, (\mathrm{w\,p})\}$ are

$$\text{valid}\ \forall \phi_{\blacksquare}\ \Box_x \phi \supset \phi \quad (37) \qquad\qquad \text{valid}\ \forall \phi_{\blacksquare}\ \Box_x \phi \supset \Box_x \Box_x \phi \quad (38)$$

As done before, we could alternatively postulate that the accessibility relations are reflexive and transitive.

Next, we encode the facts from the puzzle. For (i) we provide a persistence axiom and for (ii) an inclusion axiom:

$$\text{valid}\ \forall \phi_{\blacksquare}\ \Box_{\mathrm{p}} \Box_{\mathrm{j}} \phi \supset \Box_{\mathrm{j}} \Box_{\mathrm{p}} \phi \quad (39) \qquad\qquad \text{valid}\ \forall \phi_{\blacksquare}\ \Box_{(\mathrm{w\,p})} \phi \supset \Box_{\mathrm{p}} \phi \quad (40)$$

Finally, the facts (a)-(d) and the conclusion (e) are encoded as follows (time, place, and appointment are propositional constants, that is, constants of type $\mu \to o$ in our framework):

$$\text{valid}\ \Box_{\mathrm{p}}\, \text{time} \tag{41}$$
$$\text{valid}\ \Box_{\mathrm{p}} \Box_{\mathrm{j}}\, \text{place} \tag{42}$$
$$\text{valid}\ \Box_{(\mathrm{w\,p})} (\Box_{\mathrm{p}}\, \text{time} \supset \Box_{\mathrm{j}}\, \text{time}) \tag{43}$$
$$\text{valid}\ \Box_{\mathrm{p}} \Box_{\mathrm{j}} (\text{place} \wedge \text{time} \supset \text{appointment}) \tag{44}$$
$$\text{valid}\ \Box_{\mathrm{j}} \Box_{\mathrm{p}}\, \text{appointment} \wedge \Box_{\mathrm{p}} \Box_{\mathrm{j}}\, \text{appointment} \tag{45}$$

The combined proof problem for Example 8 is

$$(37), \ldots, (44) \models^{\mathcal{STT}} (45) \tag{46}$$

Example 8 (Wise men puzzle). Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

We employ a 4-dimensional logic combining the modalities $\Box_{\mathrm{a}}$, $\Box_{\mathrm{b}}$, and $\Box_{\mathrm{c}}$, for encoding the individual knowledge of the three wise men, and a box operator $\Box_{\mathrm{fool}}$, for encoding the knowledge that is common to all of them. The entire encoding consists now of the following axioms for $X, Y, Z \in \{a, b, c\}$ and $X \neq Y \neq Z$:

$$\text{valid } \Box_{\mathrm{fool}} ((\text{ws a}) \vee (\text{ws b}) \vee (\text{ws c})) \tag{47}$$
$$\text{valid } \Box_{\mathrm{fool}} ((\text{ws } X) \supset \Box_{Y} (\text{ws } X)) \tag{48}$$
$$\text{valid } \Box_{\mathrm{fool}} (\neg (\text{ws } X) \supset \Box_{Y} \neg (\text{ws } X)) \tag{49}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{fool}}\, \phi \supset \phi \tag{50}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{fool}}\, \phi \supset \Box_{\mathrm{fool}} \Box_{\mathrm{fool}}\, \phi \tag{51}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{fool}}\, \phi \supset \Box_{\mathrm{a}}\, \phi \tag{52}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{fool}}\, \phi \supset \Box_{\mathrm{b}}\, \phi \tag{53}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{fool}}\, \phi \supset \Box_{\mathrm{c}}\, \phi \tag{54}$$
$$\text{valid } \forall \phi .\, \neg \Box_{\mathrm{X}}\, \phi \supset \Box_{\mathrm{Y}} \neg \Box_{\mathrm{X}}\, \phi \tag{55}$$
$$\text{valid } \forall \phi .\, \Box_{\mathrm{X}}\, \phi \supset \Box_{\mathrm{Y}} \Box_{\mathrm{X}}\, \phi \tag{56}$$
$$\text{valid } \neg \Box_{\mathrm{a}} (\text{ws a}) \tag{57}$$
$$\text{valid } \neg \Box_{\mathrm{b}} (\text{ws b}) \tag{58}$$

From these assumptions we want to conclude that

$$\text{valid } \Box_{\mathrm{c}} (\text{ws c}) \tag{59}$$

Axiom (47) says that a, b, or c must have a white spot and that this information is known to everybody. Axioms (48) and (49) express that it is generally known that if someone has a white spot (or not) then the others know this. $\Box_{\mathrm{fool}}$ is axiomatized as an S4 modality in axioms (50) and (51). For $\Box_{\mathrm{a}}$, $\Box_{\mathrm{b}}$, and $\Box_{\mathrm{c}}$ it is sufficient to consider K modalities. The relation between those and common knowledge ($\Box_{\mathrm{fool}}$ modality) is axiomatized in inclusion axioms (52)–(55). Axioms (55) and (56) encode that whenever a wise man does (not) know something the others know that he does not know this. Axioms (57) and (58) say that a and b do not know whether they have a white spot. Finally, conjecture (59) states that that c knows he has a white spot. The combined proof problem for Example 7 is

$$(47), \ldots, (58) \models^{\mathcal{STT}} (59) \tag{60}$$

Example 9. A trivial example problem for the region connection calculus is (adapted from [16], p. 80):

$$
\begin{aligned}
&(\mathit{tpp}\ \text{catalunya spain}),\\
&(\mathit{ec}\ \text{spain france}),\\
&(\mathit{ntpp}\ \text{paris france}),\\
&\models^{\mathcal{STT}} (\mathit{dc}\ \text{catalunya paris}) \wedge (\mathit{dc}\ \text{spain paris})
\end{aligned}
\tag{61}
$$

The assumptions express that (i) Catalunya is a border region of Spain, (ii) Spain and France are two different countries sharing a common border, and (iii) Paris is a proper part of France. The conjecture is that (iv) Catalunya and Paris are disconnected as well as Spain and Paris.

Example 10. Within our $\mathcal{STT}$ framework we can easily put such spatial reasoning examples in an epistemic context; similar to before we distinguish between common knowledge (fool) and the knowledge of person bob and we lift the above propositions to modal propositions of type $\mu \to o$:

$$
\begin{aligned}
&\text{valid}\ \forall \phi_{\blacksquare}\ \Box_{\text{fool}}\, \phi \supset \Box_{\text{bob}}\, \phi,\\
&\text{valid}\ \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (\mathit{tpp}\ \text{catalunya spain})),\\
&\text{valid}\ \Box_{\text{fool}}\, (\lambda W_{\blacksquare} (\mathit{ec}\ \text{spain france})),\\
&\text{valid}\ \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (\mathit{ntpp}\ \text{paris france}))\\
&\models^{\mathcal{STT}}\\
&\text{valid}\ \Box_{\text{bob}}\, (\lambda W_{\blacksquare} ((\mathit{dc}\ \text{catalunya paris}) \wedge (\mathit{dc}\ \text{spain paris})))
\end{aligned}
\tag{62}
$$

We here express that (ii) from above is commonly known, while (i) and (ii) are not. (i) and (ii) are known to the educated person bob though. In this situation, conjecture (iv) still follows for bob. However, it does not follow when replacing bob by common knowledge (hence, the following problem is not provable):

$$
\ldots \models^{\mathcal{STT}} \text{valid}\ \Box_{\text{fool}}\, (\lambda W_{\blacksquare} ((\mathit{dc}\ \text{catalunya paris}) \wedge (\mathit{dc}\ \text{spain paris}))) \tag{63}
$$

6 Experiments

In our case studies, we have employed the $\mathcal{STT}$ automated reasoners LEO-II—v1.1 [10], TPS—3.080227G1d [4], IsabelleP—2009-1, IsabelleM—2009-1, and IsabelleN—2009-1.[5] These systems are available online via the SystemOnTPTP tool [28] and they support the new TPTP THF infrastructure for typed higher-order logic [11].

[5] IsabelleM and IsabelleN are model finder in the Isabelle proof assistant [25] that have been made available in batch mode, while IsabelleP applies a series of Isabelle proof tactics in batch mode.

The axiomatizations of $\mathcal{QML}^{\mathcal{STT}}$ and $\mathcal{IPL}^{\mathcal{STT}}$ are available as LCL013ˆ0.ax and LCL010ˆ0.ax in the TPTP library.[6] The example problems LCL698ˆ1.p and LCL695ˆ1.p ask about the satisfiability of these axiomatizations. Both questions are answered positively by IsabelleM and IsabelleN; IsabelleM needs 3.8 resp. 3.6 seconds and IsabelleN 3.8 resp. 3.6 seconds.

Table 1 presents the results of our experiments; the timeout was set to 120 seconds and the entries in the table are reported in seconds. Those examples which have already entered the new higher-order TPTP library are presented with their respective TPTP identifiers in the second column and the others will soon be submitted.

As expected, (31) and (63) cannot be proved by any prover and IsabelleN reports a counterexample for (31) in 34.4 seconds and for (63) in 39.7 seconds.

In summary, all but one of our example problems can be solved effectively by at least one of the reasoners. In fact, most of our example problems require only milliseconds. LEO-II solves most problems and it is the fastest prover.

7 Conclusion

The work presented in this paper has its roots in the LEO-II project (in 2006/2007 at University of Cambridge, UK) in which we first studied and employed the presented embedding of quantified multimodal logics in $\mathcal{STT}$ [8].

Our overall goal is to show that various interesting classical and non-classical logics and their combinations can be elegantly mechanized and partly automated in modern higher-order reasoning systems with the help of our logic embeddings.

Our experiments are encouraging and they provide first evidence for our claim that $\mathcal{STT}$ is suited as a framework for combining classical and non-classical logics. It is obvious, however, that $\mathcal{STT}$ reasoners should be significantly improved for fruitful application to more challenge problems in practice. The author is convinced that significant improvements — in particular for fragments of $\mathcal{STT}$ as illustrated in this paper — are possible and that they will be fostered by the new TPTP infrastructure and the new yearly higher-order CASC competitions.

Moreover, when working with our reasoners from within a proof assistant such as Isabelle/HOL the user may provide interactive help, for example, by formulating some lemmas or by splitting proof tasks in simpler subtasks.

An advantage of our approach also is that provers such as our LEO-II are generally capable of producing verifiable proof output, though much further work is needed to make these proof protocols exchangeable between systems or to explain them to humans. Finally note that it may be possible to formally verify the entire theory of our embedding(s) within a proof assistant.

Acknowledgment: The author is indebted to Geoff Sutcliffe, who, in collaboration with the author and supported by several further contributors, developed the

[6] Note that the types μ and ι are unfortunately switched in the encodings available in the TPTP: the former is used for individuals and the latter for worlds. This syntactic switch is completely unproblematic.

Problem	TPTP id	LEO-II	TPS	IsabelleP
Reasoning about Logics and Combined Logics				
(1) $\Rightarrow$ (11)	LCL699^1.p	0.0	0.3	3.6
(2) $\Rightarrow$ (12)	LCL700^1.p	0.0	0.3	13.9
(3) $\Rightarrow$ (13)	LCL701^1.p	0.0	0.3	4.0
(4) $\Rightarrow$ (14)	LCL702^1.p	0.0	0.3	15.9
(5) $\Rightarrow$ (15)	LCL703^1.p	0.1	0.3	16.0
(6) $\Rightarrow$ (16)	LCL704^1.p	0.0	0.3	3.6
(7) $\Rightarrow$ (17)	LCL705^1.p	0.1	51.2	3.9
(8) $\Rightarrow$ (18)	LCL706^1.p	0.1	0.3	3.9
(9) $\Rightarrow$ (19)	LCL707^1.p	0.1	0.3	3.6
(10) $\Rightarrow$ (20)	LCL708^1.p	0.1	0.3	4.1
(1) $\Leftarrow$ (11)	LCL709^1.p	0.0	0.3	3.7
(2) $\Leftarrow$ (12)	LCL710^1.p	—	0.3	53.8
(3) $\Leftarrow$ (13)	LCL711^1.p	0.0	0.3	3.7
(4) $\Leftarrow$ (14)	LCL712^1.p	0.0	0.3	3.8
(5) $\Leftarrow$ (15)	LCL713^1.p	—	0.8	67.0
(6) $\Leftarrow$ (16)	LCL714^1.p	1.6	0.3	29.3
(7) $\Leftarrow$ (17)	LCL715^1.p	37.9	—	—
(8) $\Leftarrow$ (18)	LCL716^1.p	—	6.6	—
(9) $\Leftarrow$ (19)	LCL717^1.p	—	—	—
(10) $\Leftarrow$ (20)	LCL718^1.p	0.1	0.4	8.1
(21)		0.1	0.4	4.3
(22)		0.2	27.4	4.0
(23)		0.1	8.9	4.0
(24)		0.1	1.2	3.7
(25)		0.1	1.7	4.2
(26)		0.2	14.8	5.4
(27)		0.1	0.6	3.7
(28)		0.2	2.3	4.0
(29)		0.1	0.9	3.9
(30)		0.1	12.8	16.5
(31)[Countersatisfiable]		—	—	—
(32)		0.0	0.3	3.6
(33)		0.0	0.3	3.6
(34)		0.1	0.4	3.6
(35)		0.2	35.5	—
(36)		0.4	—	—
Reasoning within Combined Logics				
(46)	PUZ086^1.p	0.1	—	102.4
(60)	PUZ087^1.p	0.3	—	—
(61)		2.3	—	112.7
(62)		20.4	—	—
(63)[Countersatisfiable]		—	—	—

Table 1. Performance results of $\mathcal{STT}$ provers for problems in paper.

new higher-order TPTP THF infrastructure in the EU FP7 Project THFTPTP (grant PIIF-GA-2008-219982).

References

1. Peter B. Andrews. General models and extensionality. *Journal of Symbolic Logic*, 37:395–397, 1972.
2. Peter B. Andrews. General models, descriptions, and choice in type theory. *Journal of Symbolic Logic*, 37:385–394, 1972.
3. Peter B. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof*. Kluwer Academic Publishers, second edition, 2002.
4. Peter B. Andrews and Chad E. Brown. TPS: A hybrid automatic-interactive system for developing proofs. *Journal of Applied Logic*, 4(4):367–395, 2006.
5. Matteo Baldoni. *Normal Multimodal Logics: Automatic Deduction and Logic Programming Extension*. PhD thesis, Universita degli studi di Torino, 2003.
6. Christoph Benzmüller. Automating access control logic in simple type theory with LEO-II. In Dimitris Gritzalis and Javier López, editors, *Emerging Challenges for Security, Privacy and Trust, 24th IFIP TC 11 International Information Security Conference, SEC 2009, Pafos, Cyprus, May 18-20, 2009. Proceedings*, volume 297 of *IFIP*, pages 387–398. Springer, 2009.
7. Christoph Benzmüller, Chad E. Brown, and Michael Kohlhase. Higher order semantics and extensionality. *Journal of Symbolic Logic*, 69:1027–1088, 2004.
8. Christoph Benzmüller and Lawrence Paulson. *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*, chapter Exploring Properties of Normal Multimodal Logics in Simple Type Theory with LEO-II. Studies in Logic, Mathematical Logic and Foundations. College Publications, 2008.
9. Christoph Benzmüller and Lawrence C. Paulson. *Quantified Multimodal Logics in Simple Type Theory.* SEKI Report SR–2009–02 (ISSN 1437-4447). SEKI Publications, DFKI Bremen GmbH, Safe and Secure Cognitive Systems, Cartesium, Enrique Schmidt Str. 5, D–28359 Bremen, Germany, 2009. http://arxiv.org/abs/0905.2435.
10. Christoph Benzmüller, Lawrence C. Paulson, Frank Theiss, and Arnaud Fietzke. LEO-II — A Cooperative Automatic Theorem Prover for Higher-Order Logic. In P. Baumgartner, A. Armando, and D. Gilles, editors, *Proceedings of the 4th International Joint Conference on Automated Reasoning*, number 5195 in Lecture Notes in Artificial Intelligence, pages 162–170, 2008.
11. Christoph Benzmüller, Florian Rabe, and Geoff Sutcliffe. THF0 — The Core TPTP Language for Classical Higher-Order Logic. In P. Baumgartner, A. Armando, and D. Gilles, editors, *Proceedings of the 4th International Joint Conference on Automated Reasoning*, number 5195 in Lecture Notes in Artificial Intelligence, pages 491–506, 2008.
12. Patrick Blackburn and Maarten Marx. Tableaux for quantified hybrid logic. In Uwe Egly and Christian G. Fermüller, editors, *Automated Reasoning with Analytic Tableaux and Related Methods, International Conference, TABLEAUX 2002, Copenhagen, Denmark, July 30 - August 1, 2002, Proceedings*, volume 2381 of *Lecture Notes in Computer Science*, pages 38–52. Springer, 2002.
13. Torben Braüner. Natural deduction for first-order hybrid logic. *Journal of Logic, Language and Information*, 14(2):173–198, 2005.

14. Alonzo Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.
15. Melvin Fitting. Interpolation for first order S5. *Journal of Symbolic Logic*, 67(2):621–634, 2002.
16. Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. *Many-dimensional modal logics: theory and applications*. Studies in Logic, 148. Elsevier Science, 2003.
17. Deepak Garg and Martin Abadi. A Modal Deconstruction of Access Control Logics. In R. Amadio, editor, *Proceedings of the 11th International Conference on the Foundations of Software Science and Computational Structures*, number 4962 in Lecture Notes in Computer Science, pages 216–230, 2008.
18. James Garson. Modal logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Winter 2009 edition, 2009.
19. Kurt Gödel. Eine interpretation des intuitionistischen aussagenkalküls. *Ergebnisse eines Mathematischen Kolloquiums*, 8:39–40, 1933. Also published in Gödel [20], 296–302.
20. Kurt Gödel. *Collected Works, Volume I*. Oxford University Press, 1986.
21. Robert Goldblatt. *Logics of Time and Computation*. Number 7 in Center for the Study of Language and Information - Lecture Notes. Leland Stanford Junior University, 1992.
22. Leon Henkin. Completeness in the theory of types. *Journal of Symbolic Logic*, 15:81–91, 1950.
23. Mark Kaminski and Gert Smolka. Terminating tableau systems for hybrid logic with difference and converse. *Journal of Logic, Language and Information*, 18(4):437–464, Oct 2009.
24. John C. C. McKinsey and Alfred Tarski. Some theorems about the sentential calculi of lewis and heyting. *Journal of Symbolic Logic*, 13:1–15, 1948.
25. Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL - A Proof Assistant for Higher-Order Logic*, volume 2283 of *Lecture Notes in Computer Science*. Springer, 2002.
26. David A. Randell, Zhan Cui, and Anthony G. Cohn. A spatial logic based on regions and connection. In *Proceedings 3rd International Conference on Knowledge Representation and Reasoning*, pages 165–176, 1992.
27. Krister Segerberg. Two-dimensional modal logic. *Journal of Philosophical Logic*, 2(1):77–96, 1973.
28. Geoff Sutcliffe. TPTP, TSTP, CASC, etc. In V. Diekert, M. Volkov, and A. Voronkov, editors, *Proceedings of the 2nd International Computer Science Symposium in Russia*, number 4649 in Lecture Notes in Computer Science, pages 7–23. Springer-Verlag, 2007.
29. Geoff Sutcliffe. The tptp problem library and associated infrastructure. *J. Autom. Reasoning*, 43(4):337–362, 2009.
30. Geoff Sutcliffe, Christoph Benzmüller, Chad Brown, and Frank Theiss. Progress in the development of automated theorem proving for higher-order logic. In Renate Schmidt, editor, *Automated Deduction - CADE-22, 22nd International Conference on Automated Deduction, Montreal, Canada, August 2-7, 2009. Proceedings*, volume 5663 of *LNCS*, pages 116–130. Springer, 2009.